

Anyons with anomalous gyromagnetic ratio & the Hall effect

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Abstract

Letting the mass depend on the spin-field coupling as $M^2 = m^2 - (eg/2c^2)F_{\alpha\beta}S^{\alpha\beta}$, we propose a new set of relativistic planar equations of motion for spinning anyons. Our model can accommodate any gyromagnetic ratio g and provides us with a novel version of the Bargmann-Michel-Telegdi equations in $2+1$ dimensions. The system becomes singular when the field takes a critical value, and, for $g \neq 2$, the only allowed motions are those which satisfy the Hall law. For each $g \neq 2, 0$ a secondary Hall effect arises also for another critical value of the field. The non-relativistic limit of our equations yields new models which generalize our previous “exotic” model, associated with the two-fold central extension of the planar Galilei group.

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1 Introduction

Most theoreticians argue that the gyromagnetic ratio of anyons must be $g = 2$ [1, 2, 3]. Their statement is, however, contradicted by experimentalists, who found that in a GaAs semiconductor g can be -0.44 [4]; in the Fractional Hall Effect, it can be close to zero [4, 5].

In this Letter we present a classical anyon model with arbitrary gyromagnetic ratio g . Our clue is that requiring proportionality between momentum and velocity is *not* mandatory, but a mere *assumption* that can be relaxed in a perfectly consistent manner [6, 7].

Our model, consistent with first principles, is derived in two, independent ways. Firstly, we formulate it within Souriau’s version of symplectic mechanics, equivalent to both the Lagrangian and Hamiltonian formalisms [8]. Letting the mass depend on the coupling of spin to the electromagnetic field provides us with a model valid for any g ; momentum and velocity are only parallel for $g = 2$. Our second approach follows Souriau’s *Principe de Covariance Générale*, where the equations of motion of a particle arise from the requirement of covariance w. r. t. gauge transformations [7].

For the ordinary value $g = 2$ previous results [1, 3] are recovered; new physics arises for $g \neq 2$, though. The most interesting physical application of anyons concerns indeed the Hall

effect [4, 9], which is also the main application of our model. Firstly, for a certain critical value of the field, (3.1) below, our system becomes singular and *the only allowed motions* are those which *follow the Hall law*. A *secondary Hall effect* arises for another critical value of the field, cf. (3.6). Let us insist that, in both cases, the Hall effect becomes *mandatory*.

Similar behaviour has been observed before for an “exotic” particle [10], associated with the two-fold central extension of the planar Galilei group [11] and related to noncommutative mechanics [12, 13]. The free exotic model was rederived by Jackiw and Nair (JN) as a subtle non-relativistic (NR) limit of the anyon [14]. Their clue is to relate the second extension invariant κ to relativistic spin, s , by the “magic Ansatz” [14, 15]

$$s/c^2 = \kappa. \quad (1.1)$$

Below, we extend these results to models interacting with an electromagnetic field, and present a generalized non-relativistic “exotic” particle with any g . For $g = 0$, it reduces to our previous model in [10]. Both types of Hall effects are retained in the NR limit.

2 Anomalous anyons: a model with any g

Now we present a whole family of equations valid for any value of the gyromagnetic ratio g .¹ We first recall Souriau’s group theoretical construction for the classical model which underlies geometric quantization which yields in turn the quantum representation [8]. Let us consider the neutral component of Poincaré group in $2+1$ dimensions parametrized by the $2+1$ -dimensional Minkowski space vector x^α , augmented by the three Lorentz vectors $U^\alpha, I^\alpha, J^\alpha$ such that the only nonvanishing scalar products are $U_\alpha U^\alpha = c^2$, $I_\alpha I^\alpha = J_\alpha J^\alpha = -1$. The group has two Casimirs, m and s , and a free massive spinning particle is described by the Cartan 1-form [8]

$$\alpha_0 = mU_\alpha dx^\alpha + sI_\alpha dJ^\alpha. \quad (2.1)$$

Then the classical motions are the projections onto Minkowski space of characteristic curves of the kernel of $\sigma_0 = d\alpha_0$. Minimal coupling to an external electromagnetic field amounts to adding (e times) the electromagnetic two-form $F = \frac{1}{2}F_{\alpha\beta}dx^\alpha \wedge dx^\beta$ to σ_0 ,

$$\sigma = d(mU_\alpha dx^\alpha + sI_\alpha dJ^\alpha) + \frac{1}{2}eF_{\alpha\beta}dx^\alpha \wedge dx^\beta. \quad (2.2)$$

• The spin tensor $S^{\alpha\beta} = s(I^\alpha J^\beta - I^\beta J^\alpha)$ satisfies the relation $S_{\alpha\beta}S^{\alpha\beta} = 2s^2$ and the constraint $S^{\alpha\beta}U_\beta = 0$. Therefore $S_{\alpha\beta} = s\epsilon_{\alpha\beta\gamma}U^\gamma$. Introducing the shorthand $F \cdot S = -F_{\alpha\beta}S^{\alpha\beta}$, our clue is to *replace the (constant) bare mass m in (2.2) by a mass M which depends on the electromagnetic field* [7], namely as

$$M^2 = m^2 + \frac{g}{2} \frac{eF \cdot S}{c^2} \quad (2.3)$$

provided $M^2 \geq 0$. We emphasize that our procedure is consistent with the general principles of Hamiltonian mechanics, as the two-form (2.2) is closed. Our approach is therefore equivalent to having a Lagrangian or, alternatively, a Hamiltonian framework. Let us also note that our mass formula (2.3) also yields the Bargmann-Michel-Telegdi (BMT) [16] equations in $3+1$ dimensions [7]. See also Section 4.

¹Greek indices α, β , etc. range from 0 to 2 unless otherwise specified. Latin indices i, j range from 1 to 2. We use the metric $\text{diag}(c^2, -1, -1)$.

Introducing the momentum² $p^\alpha = MU^\alpha$ and, hence, the spin tensor $S_{\alpha\beta} = (sc/\sqrt{p^2})\epsilon_{\alpha\beta\gamma}p^\gamma$ yields the Poisson brackets

$$\{x^\alpha, x^\beta\} = -\frac{1}{(p^2)^{3/2}D} S^{\alpha\beta}, \quad (2.4)$$

$$\{p_\alpha, x^\beta\} = \delta_\alpha^\beta - \frac{e}{p^2 D} F_{\alpha\gamma} S^{\gamma\beta}, \quad (2.5)$$

$$\{p_\alpha, p_\beta\} = -\frac{e}{D} F_{\alpha\beta} \quad (2.6)$$

where we put $D = 1 + eF \cdot S/2p^2$. Our system is regular provided $D \neq 0$.

• The two-form σ lives indeed on the unit-tangent bundle, $U_\alpha U^\alpha = c^2$, of 2 + 1-dimensional Minkowski space. The momenta are coordinates on this bundle, which can be viewed hence as the 5-dimensional surface sitting in 6-dimensional phase space defined by the constraint

$$p^2 = M^2 c^2 \quad (2.7)$$

with the mass M given in (2.3). Assuming, for simplicity, that the electromagnetic field is constant, a straightforward calculation shows that our restricted two-form reads

$$\sigma = dp_\alpha \wedge dx^\alpha + \frac{(s/c^2)}{2M^3} \epsilon_{\alpha\beta\gamma} p^\alpha dp^\beta \wedge dp^\gamma + \frac{1}{2} e F_{\alpha\beta} dx^\alpha \wedge dx^\beta. \quad (2.8)$$

Working with the Hamiltonian system given by the Poisson brackets (2.4-2.5-2.6) and the Hamiltonian $H = p^2 - M^2 c^2$ is equivalent to finding the kernel of the closed two-form σ in (2.8). Generically, i. e. when

$$D = 1 + \frac{eF \cdot S}{2M^2 c^2} \quad (2.9)$$

does not vanish, the kernel is 1-dimensional, spanned by δx^α and δp^α such that³

$$D \delta x^\alpha = \frac{p_\beta \delta x^\beta}{M c^2} \left[G \frac{p^\alpha}{M} - \frac{es}{2M^2} \left(1 - \frac{g}{2} \right) \epsilon^{\alpha\beta\gamma} F_{\beta\gamma} \right], \quad \delta p^\alpha = e F_\beta^\alpha \delta x^\beta$$

where

$$G = 1 + \frac{g}{2} \cdot \frac{eF \cdot S}{2M^2 c^2}. \quad (2.10)$$

Let us first assume that neither of the factors D and G vanishes. The integral curves of $\ker(\sigma)$ are conveniently parametrized by τ such that $\delta\tau = p_\alpha \delta x^\alpha / M c^2$ and identified with time. Then, we end up with

$$D \frac{dx^\alpha}{d\tau} = G \frac{p^\alpha}{M} + (g-2) \frac{es}{4M^2} \epsilon^{\alpha\beta\gamma} F_{\beta\gamma}, \quad (2.11)$$

$$\frac{dp^\alpha}{d\tau} = e F_\beta^\alpha \frac{dx^\beta}{d\tau}. \quad (2.12)$$

These are the new equations of motion we propose for a relativistic particle with spin and magnetic moment (we identify with anomalous anyons), moving in the plane in a constant external electromagnetic field. The Lorentz equation retains its usual form and it is only the relation between the velocity and the momentum, (2.11), which is modified. In general, the motion depends also on the spin.

²We stress that our U_α is the (normalized) *momentum* and *not* the velocity, see below.

³The complicated form of the coefficients here, and also in (5.2) of Section 5, is due to the particular form of the mass relation (2.3), see Eq. (8.1) in the Conclusion.

Let us analyse our equations (2.11-2.12) in some detail.

- In the absence of an external field, our construction reduces to that of Souriau [8], and we recover the free spinning anyon [2].

- Contracting (2.11) by the field $F_{\alpha\beta}$ and using the Lorentz equation (2.12) yields furthermore that the spin-field dependent mass, M in (2.3), is a *constant of the motion*,

$$\frac{dM}{d\tau} = 0. \quad (2.13)$$

- For $g = 2$ the term proportional to $g - 2$ in (2.11) drops out, leaving us with the spin-independent equation

$$\frac{dx^\alpha}{d\tau} = \frac{p^\alpha}{M}. \quad (2.14)$$

It follows that our parameter τ is now proper time, since $(dx/d\tau)^2 = c^2$. Redefining time⁴ according to $\lambda = (m/M)\tau$ transforms our equations into the form posited by Chou et al. [1],

$$\frac{dx^\alpha}{d\lambda} = \frac{p^\alpha}{m}, \quad \frac{dp^\alpha}{d\lambda} = \frac{e}{m} F^\alpha_\beta p^\beta. \quad (2.15)$$

These equations are associated with the two-form (2.8), where M is our (2.3) with $g = 2$, and the Hamiltonian

$$H = \frac{1}{2m}(p^2 - m^2 c^2) - \frac{e}{2m} F \cdot S = \frac{1}{2m}(p^2 - M^2 c^2). \quad (2.16)$$

This latter is chosen so as to cancel the effect of the spin term in the two-form and to enforce the relation (2.15) posited between p^α and $dx^\alpha/d\lambda$.

- The new feature of our equations (2.11-2.12) is that for $g \neq 2$ momentum and velocity are no longer parallel. It follows that our spin constraint is in general different from $S_{\alpha\beta} dx^\beta/d\tau = 0$, which is also used sometimes.

- The general equations of motion (2.11-2.12) are highly nonlinear in the field strength F . Linearizing up to higher-order terms in the quantity $\frac{eF \cdot S}{m^2 c^2} \ll 1$, we have $M \cong m$ where \cong means “up to higher order terms in the field F ”. We end up with the novel relativistic planar BMT-type anyon equations

$$\begin{aligned} \frac{dx^\alpha}{d\tau} &\cong \frac{p^\alpha}{m} - \frac{es}{2m^2} \left(1 - \frac{g}{2}\right) \epsilon^{\alpha\beta\gamma} F_{\beta\gamma}, \\ \frac{dp^\alpha}{d\tau} &= e F^\alpha_\beta \frac{dx^\beta}{d\tau}. \end{aligned} \quad (2.17)$$

For $g = 2$ we recover the equations (2.15).

3 Relativistic Hall effects

Returning to the general case the system becomes singular when the factor D in (2.9) vanishes,

$$\frac{eF \cdot S}{2M^2 c^2} = -1. \quad (3.1)$$

Then comparison with (2.3) shows that the critical mass M' is given by

$$(M')^2 = \frac{m^2}{1+g}. \quad (3.2)$$

⁴Time redefinition changes the gyromagnetic factor, confirming that $g = 2$ can be viewed as a gauge condition [17], namely that of world line reparametrization.

Hence $(M')^2 > 0$ whenever $g > -1$ that we shall henceforth require. Then the velocity is eliminated from the l.h.s. of (2.11) leaving us with

$$\left(1 - \frac{g}{2}\right) \left(p^\alpha - \frac{es}{2M'} \epsilon^{\alpha\beta\gamma} F_{\beta\gamma}\right) = 0. \quad (3.3)$$

- In the “normal” case, $g = 2$, the equation is identically satisfied. Then $D = G \neq 0$ drops out from (2.11) before taking the $D \rightarrow 0$ limit, and (2.15) holds true therefore even when $D = G \rightarrow 0$, despite the fact that the closed two-form σ becomes singular.

- In the anomalous case $g \neq 2$, however, (3.3) allows us to infer that

$$p^\alpha = \frac{es}{2M'} \epsilon^{\alpha\beta\gamma} F_{\beta\gamma}. \quad (3.4)$$

Note that $G = 1 - g/2 \neq 0$. Hence $p^0 = (es/M'c^2)B$ and $p^i = \epsilon^{ij}(es/M'c^2)E_j$. Then (3.4) implies that $\dot{p}^\alpha = 0$ since the field is constant. Hence, by (2.12) and $\det F = 0$, we readily obtain $\dot{x}^\alpha = dx^\alpha/d\tau \propto \epsilon^{\alpha\beta\gamma} F_{\beta\gamma} \propto p^\alpha$. The velocity $v^i = p^i/p^0$ satisfies therefore the Hall law

$$v^i = \epsilon^{ij} \frac{E_j}{B}. \quad (3.5)$$

Remarkably, a *secondary Hall effect* can also arise. Let us indeed require that the coefficient of the momentum on the r.h.s. of (2.11) vanishes, $G = 0$, i.e.

$$\frac{eF \cdot S}{2M^2 c^2} = -\frac{2}{g}. \quad (3.6)$$

Then $D = 1 - 2/g \neq 0$ and the system is regular. The squared mass,

$$(M'')^2 = \frac{1}{3}m^2, \quad (3.7)$$

is always positive. The velocity will be again determined by the electromagnetic field alone, namely according to

$$\dot{x}^\alpha = \frac{3g}{4} \cdot \frac{es}{m^2} \epsilon^{\alpha\beta\gamma} F_{\beta\gamma}. \quad (3.8)$$

Then $v^i = \dot{x}^i/\dot{x}^0$ satisfies once again the Hall law (3.5)! Let us observe that the momentum has been decoupled, and can be determined by solving (2.12).

Note for further reference that both critical conditions (3.1) and (3.6) link the fields and the spin, see Section 6.

4 The origin of the mass formula (2.3)

Our generalized model relies on the mass formula (2.3). Its origin can be explained from a rather different viewpoint. Some time ago [6] a set of equations of motion for a general relativistic spinning particle in a gravitational and electromagnetic field has been proposed. These latter, called the Mathisson-Weyssenhoff-Papapetrou equations, read

$$\dot{p}^\alpha = eF^\alpha_\beta \dot{x}^\beta - \frac{1}{2}R(S)^\alpha_\beta \dot{x}^\beta + \frac{1}{2}M^{\beta\gamma} \nabla^\alpha F_{\beta\gamma}, \quad (4.1)$$

$$\dot{S}^{\alpha\beta} = p^\alpha \dot{x}^\beta - p^\beta \dot{x}^\alpha - M^\alpha_\gamma F^{\gamma\beta} + M^\beta_\gamma F^{\gamma\alpha}, \quad (4.2)$$

supplemented by the conservation law $\dot{e} = 0$. Here ∇ is the Levi-Civita connection of the metric and $R(S)^\alpha_\beta = R^\alpha_{\mu\nu\beta} S^{\mu\nu}$, where $R^\alpha_{\mu\nu\beta}$ is the Riemann curvature. [We use the convention

$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)v^\alpha = R^\alpha_{\mu\nu\beta}v^\beta$.] The quantities p^α , $S^{\alpha\beta}$, e and $M^{\alpha\beta}$ here are interpreted as the linear momentum, the (skew-symmetric) spin tensor, the electric charge, and the electromagnetic dipole moment, respectively. In this paper we only consider the flat case.

Equations (4.1-4.2) can be derived from the requirement of gauge invariance (Souriau's "*Principe de covariance générale*") of the theory alone [7]. They are universal in that they hold independently of the relation between momentum and velocity. To get a deterministic system, this latter has to be specified by supplementary constraints [7].

Firstly, to guarantee the localizability of the particle, we require $S^{\alpha\beta}p_\beta = 0$. Our particle should moreover carry no electric dipole moment; this is expressed as $M^{\alpha\beta} = \chi S^{\alpha\beta}$ where χ is some function, identified as the scalar magnetic moment. These conditions actually make the system deterministic. Let us show that they also yield some unexpected result related to the magnetization energy, (4.3) below.

It is straightforward to prove that the scalar spin s , defined by $s^2 = \frac{1}{2} S_{\alpha\beta} S^{\alpha\beta}$ is a constant of the motion, $\dot{s} = 0$. In the sequel we promote s to a constant of the system. One thus finds that $\dot{p}_\alpha \dot{S}^{\alpha\beta} p_\beta = 0$; then (4.2) yields

$$p_\alpha \dot{p}^\alpha p_\beta \dot{x}^\beta - p_\alpha p^\alpha \dot{p}_\beta \dot{x}^\beta - \chi \dot{p}_\alpha S^{\alpha\beta} F_{\beta\gamma} p^\gamma = 0$$

and (4.1) enables us to write the latter equation as

$$\frac{1}{2}(p^2)^\cdot p_\beta \dot{x}^\beta - \frac{1}{2}p^2 M^{\alpha\beta} \dot{F}_{\alpha\beta} + \chi p_\alpha \dot{S}^{\alpha\beta} F_{\beta\gamma} p^\gamma = 0.$$

Some more work allows us to show that

$$(p^2)^\cdot p_\beta \dot{x}^\beta - \chi p^2 S^{\alpha\beta} \dot{F}_{\alpha\beta} + 2\chi p^2 \dot{x}_\alpha F^{\alpha\beta} p_\beta = 0.$$

Since $F_{\alpha\beta} \dot{S}^{\alpha\beta} = -2\dot{x}_\alpha F^{\alpha\beta} p_\beta$, we end up with $(p^2)^\cdot p_\beta \dot{x}^\beta = \chi p^2 (F_{\alpha\beta} S^{\alpha\beta})^\cdot$. The latter equation is consistent with the mass being given by an otherwise *arbitrary* function of the spin-field coupling, viz.

$$p^2 = M^2 c^2 \quad \text{where} \quad M = M(\phi), \quad \text{and} \quad \phi = eF \cdot S. \quad (4.3)$$

Let us insist that all these results hold true for any dimension of spacetime. In 3+1 dimensions, a similar procedure would yield the original BMT equations [16], supplemented with a modified velocity-momentum relation [7].

• Further justification of our key formula (2.3) is obtained for spin 1/2 field with $g = 2$ by considering the Dirac equation

$$(iD_\alpha \gamma^\alpha - mc)\Psi = 0, \quad (4.4)$$

where $D_\alpha = \partial_\alpha - ieA_\alpha$ is the gauge-covariant derivative. Then applying the conjugate operator on the l.h.s. yields

$$(D_\alpha D^\alpha + M^2 c^2)\Psi = 0, \quad M^2 = m^2 + \frac{eF \cdot S}{c^2} \quad (4.5)$$

which is clearly consistent with (2.7) and (2.3) with $g = 2$. Extension to any g is considered in the third reference of [7].

5 The non-relativistic limit

Let us consider, at last, the non-relativistic limit of the general system (2.11-2.12). We shall use

$$F \cdot S = \frac{2s}{mc^2} \left(-\epsilon^{ij} p_i E_j - p^0 B \right) \approx -2sB$$

where \approx stands for “up to higher order terms in c^{-2} ”, along with the generalized Jackiw-Nair-type Ansatz [14, 15]

$$s = \theta m^2 c^2 + s_0. \quad (5.1)$$

In the NR limit

$$\begin{aligned} M^2 &\approx M_{NR}^2 = m^2(1 - g\theta eB), \\ D &\approx D_{NR} = \frac{1 - (g+1)\theta eB}{1 - g\theta eB}, \\ G &\approx G_{NR} = \frac{1 - (3g/2)\theta eB}{1 - g\theta eB} \end{aligned} \quad (5.2)$$

provided $g\theta eB \neq 1$. Using $p^0 \approx M$, the time component of (2.11) yields $\dot{x}^0 \approx 1$ so that τ becomes nonrelativistic time. The equations of motion reduce therefore to

$$m(1 - (g+1)\theta eB)\dot{x}^i = \left(1 - (3g/2)\theta eB\right)\frac{p^i}{M_{NR}} - \left(1 - \frac{g}{2}\right)m\theta\epsilon^{ij}E_j \quad (5.3)$$

$$\dot{p}^i = eE^i + eB\epsilon^{ij}\dot{x}_j \quad (5.4)$$

where the dot means now derivation w.r.t. nonrelativistic time. For $g\theta eB \rightarrow 1$ the system would blow up.

- For $g = 0$ we recover those equations written in Ref. [10].
- For $g = 2$ both coefficients $D_{NR} = G_{NR}$ drop out as long as $\theta eB \neq 1/3$ [for which it would reduce to $0 = 0$]. Then velocity and momentum become parallel

$$m\dot{x}^i = \frac{1}{\sqrt{1 - 2\theta eB}}p^i \quad (5.5)$$

($\theta eB \neq 1/2$). Note that (5.5) is also the NR limit of (2.14). When $\theta eB \rightarrow 1/2$, (2.14) has no NR limit since $M^2 \rightarrow 0$.

Let us observe that the two relativistic invariants m and s , interpreted as relativistic mass and spin, respectively, give rise, in the NR limit, to *two pairs of nonrelativistic invariants*, namely non-relativistic mass and internal energy, and non-relativistic spin and exotic structure, respectively. Equation (5.1) actually defines the non-commutative parameter $\theta = \kappa/m^2$, and s_0 is interpreted as nonrelativistic spin [15].

6 Non-relativistic Hall effects

Let us now consider the critical cases in the NR limit for $g \neq 2$.

- The coefficient D_{NR} of \dot{x} on the l.h.s. of (5.3) vanishes when

$$B' = \frac{1}{1+g} \cdot \frac{1}{e\theta} \quad (6.1)$$

[which is just the NR limit of the first critical condition (3.1)]. Then the Hall law, (3.5) is satisfied. (Alternatively, the NR limit of (3.4) is $p^i = e(\kappa/M')\epsilon^{ij}E_j$ and $p^0 = e(\kappa/M')B$.) Equation (6.1) generalizes the result found in [10].

The second critical case $G_{NR} = 0$ [which is also the NR limit of (3.6)] requires

$$B'' = \frac{2}{3g} \cdot \frac{1}{e\theta}, \quad (6.2)$$

as long as $g \neq 0$. Insertion into (5.3) yields again the Hall law (3.5). Alternatively, the NR limit of equation (3.8) provides us with the same conclusion.

When $g = 2$ no Hall effect arises, since $\theta eB \rightarrow 1/2$ is inconsistent, and $\theta eB \rightarrow 1/3$ is already regular.

7 NR equations in the weak-field limit

Further insight is gained by studying the weak-field limit of the equations (5.3-5.4). If both $g\theta eB \ll 1$ and $\theta eB \ll 1$, we can neglect higher-order terms in the field and readily obtain $M_{NR} \cong m(1 - (g/2)\theta eB)$. When $1 - g\theta eB \neq 0$, the weak field limit of our equations (5.3-5.4) retains the form

$$\begin{aligned} m^* \dot{x}^i &\cong p^i - \left(1 - \frac{g}{2}\right) m\theta \epsilon^{ij} eE_j, \\ \dot{p}^i &\cong eE^i + eB\epsilon^{ij} \dot{x}_j. \end{aligned} \quad (7.1)$$

where

$$m^* = m(1 - \theta eB) \quad (7.2)$$

is the effective mass introduced in [10]. These are our new, non-relativistic, “exotic BMT” equations valid in a weak electromagnetic field for any g . They can also be obtained taking the NR limit of the weak-field relativistic equations (2.17).

- For $g = 2$ we find

$$m^* \dot{x}^i \cong p^i \quad (7.3)$$

supplemented with the Lorentz equation $\dot{p}^i \cong eE^i + eB\epsilon^{ij} \dot{x}_j$, which is in fact the NR limit of the system (2.15). To find this limit one has to use $p^2 = M^2 c^2$ instead of the naive condition $p^2 = m^2 c^2$, which is inconsistent with the model. This is the only case when velocity and momentum are parallel.

- When the gyromagnetic ratio *vanishes*, viz. $g = 0$, equations (7.1) reduce to the “exotic” equations of motion discovered in [10]. The latter are hence *not* the NR limit of the model in [1], cf. [14].

- For a generic gyromagnetic factor, g , equations (7.1) describe the motions of charged nonrelativistic particles in the plane, endowed with both anomalous magnetic moment and “exotic” structure, given by the non-commutative parameter θ (alias Galilei invariant κ). A look at (7.1) shows that the gyromagnetic ratio can only be detected if $\theta \neq 0$. This is not a surprise, if we remember that by (1.1) the “exotic” structure is a “nonrelativistic shadow” of relativistic spin. The equations (7.1) are Hamiltonian, with

$$\omega = dp_i \wedge dx^i + \frac{eB}{2} \epsilon_{ij} dx^i \wedge dx^j + \frac{\Theta}{2} \epsilon^{ij} dp_i \wedge dp_j, \quad h = \frac{\vec{p}^2}{2\tilde{m}} + eV \quad (7.4)$$

where V is the electric potential, and

$$\tilde{m} = m \left(1 - \frac{g}{2} \theta eB\right), \quad \Theta = \frac{1 - g/2}{1 - (g/2)\theta eB} \theta \quad (7.5)$$

provided $1 - (g/2)\theta eB \neq 0$. For *any* $g \neq 2$, we recover hence our previously introduced “exotic” system in [10] with redefined parameters \tilde{m} and Θ . Interestingly, the effective mass remains unchanged, $\tilde{m}^* = \tilde{m}(1 - e\Theta B) = m^*$. The Poisson brackets of the coordinates associated with the (singular) symplectic structure in (7.4),

$$\{x_i, x_j\} = \frac{\tilde{m}}{m^*} \Theta \epsilon_{ij} = \left(1 - \frac{g}{2}\right) \frac{m}{m^*} \theta \epsilon_{ij}, \quad (7.6)$$

are nonvanishing except for $g = 2$, when the system becomes commutative and reduces to the usual “non-exotic” particle in an electromagnetic field.

- Our weak-field approximation would again accommodate both types of Hall effects, with some modified critical field values. These are, however, *not* physical since the critical values are *not* weak, but rather fixed by the conditions (6.1) and (6.2). But for these values our weak-field derivation given for (7.1) becomes inconsistent.

8 Conclusion and outlook

Our generalized anyon model with any gyromagnetic ratio g relies on the mass formula (2.3), which for $g \neq 2$ lifts the conventional requirement that velocity and momentum should be parallel. A justification comes the Mathisson-Weyssenhoff-Papapetrou equations, also derived from Souriau’s *covariance générale* [6, 7]. It is worth mentioning that our mass formula (2.3) is just one possibility, convenient in a weak electromagnetic field. Other choices have also been considered [7, 18]. A general mass function $M(\phi)$, see (4.3), would generalize (2.11) to

$$\left(1 + \frac{eF \cdot S}{2M^2 c^2}\right) \frac{dx^\alpha}{d\tau} = \left(1 + \frac{eF \cdot S}{M} \frac{dM}{d\phi}\right) \frac{p^\alpha}{M} - \frac{es}{2M^2} \left(1 - 2c^2 M \frac{dM}{d\phi}\right) \epsilon^{\alpha\beta\gamma} F_{\beta\gamma} \quad (8.1)$$

which is (5.3), with gyromagnetic factor

$$g = 4c^2 M \frac{dM}{d\phi}. \quad (8.2)$$

Again, when the system becomes singular, cf. (3.1), or when the momentum is decoupled, cf. (3.6), all motions obey the Hall law, provided $g \neq 2$.

The “Jackiw-Nair” limit of our model provides us with a non-relativistic model, (7.1) for any g . In the ordinary case $g = 2$ one gets a commutative theory. For $g \neq 2$ the NR limits of (3.1) and (3.6) yield two types of critical values, (6.1) and (6.2), respectively.

What is the physical interpretation our two types of Hall effects ? We do not have a definitive answer as yet. A hint may come from the weak-field, NR picture, though. Since for all $g \neq 2$ the system can be brought into the same form, namely that of [10], it follows that quantization of the primary critical case yields the Laughlin description of the FQHE [9]. In particular, all wave functions belong to the lowest Landau level [10, 13]. The first type of effect generalizes the one in [10] to any g . The second type effect is new, and is still somewhat mysterious; it is related to a spontaneous decoupling of momentum.

But is $g \neq 2$ possible at all ? The strategy of [1], for example, to prove that $g = 2$, is to *posit* that anyons in an external electromagnetic field satisfy the usual Lorentz equations, (2.15). The latter are only consistent with the 3+1-dimensional BMT equations [16] when $g = 2$. The same statement remains true for us : consistency of our general planar model with either the original [16], or the suitably modified [7] BMT system requires $g = 2$.

Other physical instances of singling out $g = 2$, including unitarity in $3 + 1$ dimensional gauge theory, string theory, as well as some extra gauge symmetry [17] or supersymmetry [19], are known. Do these arguments force us to discard our equations (2.11-2.12) for $g \neq 2$? We argue that *no*: consistency of the planar and the spatial systems may *not* be mandatory — just like it is impossible to deduce fractional spin from a $3 + 1$ -dimensional model with half-integer spin! These are the peculiar properties of planar physics that allow for anomalous anyons. Hence, there is no reason to discard our theory with an arbitrary g , as long as we consider $2 + 1$ dimensions as physical. Similarly, while supersymmetry may be a useful property, it can not be viewed as a fundamental physical requirement.

What is the experimental situation? Band effects in a semiconductor renormalize the electron mass and gyromagnetic factor. The band mass in GaAs is, for example, considerably smaller (typically a few percent) than the electron mass. Similarly, it is argued that the gyromagnetic factor in a semiconductor is determined by the spin-orbit coupling [4, 5]. These facts appear to be, at least, not inconsistent with our ideas expressed here: the small mass reminds one of our vanishing effective mass condition, $m^* = 0$ in [10]. The latter model has $g = 0$.

Anyons have long been thought to play a fundamental role to explain the (Fractional) Quantum Hall Effect; to our knowledge, this is in fact the only physical instance where anyons have experimentally been detected [20]. We believe, therefore, that the Hall effect(s), becoming *mandatory* for some critical value(s) of our parameters, provide us with a strong argument in favor of the physical reality of anomalous anyons in general, and for our theory in particular.

At last, in $3 + 1$ dimensions, similar ideas were put forward by Dixon [6] and developed in the seventies [7, 18]. Previous work of Skagerstam and Stern [19] espouses, in a Lagrangian framework, a viewpoint similar to ours here. For $g = 2$, our commutation relations (2.4)-(2.5)-(2.6) can be seen, when taking into account our mass-shell condition (2.3), to agree with those, # (26), in [1]. The difference comes precisely from our choosing (2.3), while the Authors of [1] posit the simple, spin-independent Lorentz equations (2.15) (that imply $g = 2$).

The elaboration of the planar case and its application to the Hall effect are, to our knowledge, new.

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References

- [1] C. Chou, V. P. Nair and A. Polychronakos, *On the electromagnetic interaction of anyons. Phys. Lett.* **B304**, 105 (1993).
- [2] B.-S. Skagerstam and A. Stern, *Topological quantum mechanics in $2 + 1$ dimensions*, *Int. Journ. Mod. Phys.* **A 5**, 1575 (1990); M. S. Plyushchay, *Relativistic model of anyons. Phys. Lett.* **B248**, 107 (1990); R. Jackiw and V. P. Nair, *Relativistic wave equation for anyons. Phys. Rev.* **D43**, 1933 (1991); S. Ghosh, *Spinning particles in $2 + 1$ dimensions. Phys. Lett.* **B338**, 235 (1994).
- [3] J. L. Cortés, J. Gamboa, and L. Velázquez, *Electromagnetic interaction of anyons in non-relativistic quantum field theory. Int. Journ. Mod. Phys.* **A9**, 953 (1994); S. Ghosh, *Anyons*

- in an electromagnetic field and the Bargmann-Michel-Telegdi equation. Phys. Rev. D* **51**, 5827 (1995); J. L. Cortés, and M. S. Plyushchay, *Anyons as spinning particles. Int. J. Mod. Phys. A* **11**, 3331 (1996).
- [4] F. Ezawa, *The Quantum Hall Effects. Field theoretical approach and related topics*. World Scientific: Singapore (2000). p. 143.
- [5] D. K. Maude et al. *Spin excitations of a two-dimensional electron gas in the limit of vanishing Landé g factor. Phys. Rev. Lett.* **77**, 4604 (1996); D. R. Leadley, et al. *Fractional Quantum Hall Effect measurements at zero g factor. Phys. Rev. Lett.* **79**, 4246 (1997).
- [6] M. Mathisson, *Das zitternde elektron und seine Dynamik. Acta Phys. Pol.* **6**, 218 (1937); J. Weyssenhoff and A. Raabe, *Relativistic dynamics of spin-fluids and spin-particles. Acta Physica Polonica* **9**, 7, (1947); A. Papapetrou, *Proc. Roy. Soc. A* **209**, 248-258 (1951); W. G. Dixon, *On a classical theory of charged particles with spin and the classical limit of the Dirac equation. Il Nuovo Cimento* **38**, 1616 (1965).
- [7] Ch. Duval, H. H. Fliche and J.-M. Souriau, *Un modèle de particule à spin dans le champ gravitationnel et électromagnétique. Comptes Rendus de l'Académie des Sciences de Paris, Série A.* **274**, 1082 (1972); J.-M. Souriau, *Modèle de particule à spin dans le champ électromagnétique et gravitationnel, Ann. Inst. Henri Poincaré*, **20 A**, 315 (1974). Ch. Duval, *The general relativistic Dirac-Pauli particle: an underlying classical model, Ann. Inst. Henri Poincaré*, **25 A**, 345 (1976). See also Ch. Duval, *Un modèle de particule à spin dans le champ électromagnétique et gravitationnel extérieur. Thèse de 3e cycle. Marseille* (1972); S. Sternberg, in *Proc. 2nd Bonn Conf. Diff. Geom. Meths. in Math. Phys. Springer Lecture Notes in Mathematics* **676** p. 1-80 (1978).
- [8] J.-M. Souriau, *Structure des systèmes dynamiques*, Dunod: Paris (1970); *Structure of Dynamical Systems: a Symplectic View of Physics*. Birkhäuser (1997).
- [9] *Quantum Hall Effect*. Ed. M. Stone, World Scientific Singapore (1992).
- [10] C. Duval and P. A. Horváthy, *The exotic Galilei group and the "Peierls substitution". Phys. Lett. B* **479**, 284 (2000) [[hep-th/0002233](#)]; *Exotic Galilean symmetry in the non-commutative plane, and the Hall effect. Journ. Phys. A* **34**, 10097 (2001).
- [11] J.-M. Lévy-Leblond, *Galilei group and Galilean invariance. in Group Theory and Applications* (Loebl Ed.), **II**, Acad. Press, New York, p. 222 (1972); A. Ballesteros, N. Gadella and M. del Olmo, *Moyal quantization of 2+1 dimensional Galilean systems. Journ. Math. Phys.* **33**, 3379 (1992); Y. Brihaye, C. Gónzalez, S. Giller and P. Kosiński, *Galilean invariance in 2+1 dimensions. hep-th/9503046* (unpublished); D. R. Grigore, *Transitive symplectic manifolds in 1 + 2 dimensions. Journ. Math. Phys.* **37**, 240 (1996), and *The projective unitary irreducible representations of the Galilei group in 1 + 2 dimensions. ibid.* **37** 460 (1996). J. Lukierski, P. C. Stichel, W. J. Zakrzewski, *Galilean-invariant 2 + 1-dimensional models with a Chern-Simons-like term and $d = 2$ noncommutative geometry. Annals of Physics (N. Y.)* **260**, 224 (1997).

- [12] See, e. g., V. P. Nair and A. P. Polychronakos, *Quantum mechanics on the non-commutative plane and sphere*. *Phys. Lett.* **B505**, 267 (2001); J. Gamboa, M. Loewe, and J. C. Rojas, *Non-commutative Quantum Mechanics*. *Phys. Rev.* **D 64**, 067901 (2001).
- [13] P. A. Horváthy, *The non-commutative Landau problem* *Ann. Phys. (N. Y.)* **299**, 128-140 (2002), [hep-th/0201007].
- [14] R. Jackiw and V. P. Nair, *Anyon spin and the exotic central extension of the planar Galilei group*. *Phys. Lett.* **B 480**, 237 (2000) [hep-th/0003130]. See also [15] below.
- [15] C. Duval and P. A. Horváthy, *Spin and exotic Galilean symmetry*. *Phys. Lett.* **B 547** 306-312 (2002) [hep-th/0209166].
- [16] V. Bargmann, L. Michel, and V. L. Telegdi, *Precession of the polarization of the particles moving in a homogeneous electromagnetic field*. *Phys. Rev. Lett.* **2**, 435 (1959).
- [17] R. Jackiw, *$g = 2$ as a gauge condition*, *Phys. Rev.* **D 57**, 2635 (1998).
- [18] H. P. Künzle, *Canonical dynamics of spinning particles in gravitational and electromagnetic fields*. *J. Math. Phys.* **13**, 739 (1972).
- [19] B. S. Skagerstam and A. Stern, *Lagrangian descriptions of classical charged particles with spin*. *Physica Scripta* **24**, 493 (1981).
- [20] D. Sen (private communication).